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LETTER TO THE EDITOR

Symmetries of dynamical systems and convergent normal forms

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Abstract. It is shown that under suitable conditions involving, in particular, the existence of analytic constants of motion, the presence of Lie point symmetries can ensure the convergence of the transformation taking a vector field (or dynamical system) into normal form.

The classical technique of transforming a given vector field (describing, for example, the flow of a dynamical system (DS)) into normal form (NF) (in the sense of Poincaré and Dulac) is a well known and useful method of investigation [1, 2] (see also, e.g., [3] for further references); it is also well known, however, that Poincaré–Dulac series are, in general, only formal or asymptotic series. The convergence of a normalizing transformation is in fact a quite 'rare' event, and often one considers truncated series (and then approximate transformations (see, e.g., [4, 5] and references therein)). In this letter we want to propose some results (details and complete proofs will be presented in a separate paper) concerning the convergence of the normalizing transformations: precisely, we will show that under suitable conditions the presence of some Lie symmetry [6, 7] of the vector field can ensure the convergence of the normalizing transformation. As a particular case, we recover a remarkable result which has recently been obtained by Bruno and Walcher for two-dimensional problems [8]; in the same context, see also [9] for an older result, the proof of which, however, has been recognized to be incomplete [10].

We will consider n-dimensional vector fields f in the form of DS

$$\dot{u} = f(u) = Au + F(u) \qquad u = u(t) \in R^n \tag{1}$$

where $\dot{u} = \mathrm{d}u/\mathrm{d}t$, f is assumed to be analytic in a neighbourhood of $u = 0$, with $f(0) = 0$, and where the matrix $A \equiv (\nabla f)(0)$ is assumed to be non-zero and diagonalizable (see [3] for a discussion on the non-diagonalizable case). We introduce the usual notion of Lie–Poisson brackets

$$\{f, g\}_k = (f \cdot \nabla) g_k - (g \cdot \nabla) f_k \qquad (k = 1, \ldots, n) \tag{2}$$

together with that of a 'homological operator' $\mathcal{A}$ associated with the matrix A

$$\mathcal{A}(f) = \{Au, f\} = (Au) \cdot \nabla f - Af. \tag{3}$$

It is known that a nonlinear vector function $h = h(u)$ is said to be in NF with respect to A (or resonant with A) if

$$\mathcal{A}(h) = 0. \tag{4}$$

Let us also recall the following basic theorem [2]: a DS (1) can be put into NF by means of a convergent transformation if it fulfils the following two conditions:

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Condition A. There is a coordinate transformation changing f to $\tilde{f}$, where $\tilde{f}$ has the form

$$\tilde{f} = Au + \alpha(u)Au$$

and $\alpha(u)$ is some scalar-valued power series (with $\alpha(0) = 0$).

Condition ω. Let $\omega_k = \min |(q, a)|$ (where a_i are the eigenvalues of A, and parentheses stand for the scalar product) for all positive integers q_i such that $\sum_{i=1}^{n} q_i < 2^k$ and $(q, a) \neq 0$, then the series

$$\sum_{k=1}^{\infty} 2^{-k} \ln \omega_k$$

is convergent.

While condition ω is a weak condition, controlling the appearance of small divisors [2], condition A is clearly a rather strong restriction. We explicitly assume that all the DSs considered in the remainder of this paper will satisfy condition ω, but do *not* satisfy condition A.

A vector function

$$g(u) = Bu + G(u) \tag{5}$$

(not proportional to f) is said to be a Lie point (time independent) symmetry for the DS (1) if

$$\{f, g\} = 0. \tag{6}$$

In terms of Lie algebras, one would say that the vector field operator $g \cdot \nabla$ generates a symmetry of the DS.

We can now state our results.

Theorem 1. Assume that

(i) the DS (1) admits an analytic symmetry (5) where either the matrix B is proportional to A, or $B = 0$ and $G(u)$ is not proportional to $F(u)$;

(ii) once in NF, the DS takes the form

$$\dot{u} = h(u) = Au + \alpha(u)Au + \mu(u)Mu \tag{7}$$

where M is some matrix (not proportional to A), α and μ some scalar functions, and the two linear problems $\dot{u} = Au$ and $\dot{u} = Mu$ do not admit time-independent common constants of motion.

Then, the DS can be put in NF by means of a convergent normalizing transformation.

Sketch of the proof. If $B = 0$, consider the new symmetry $g+f = Au+F+G$ and consider that, due to (4), the linear field $g_A = Au$ is a symmetry for the NF. Using hypothesis (ii), one can show (see [11]) that the NF (7) does not admit constants of motion (i.e. functions $\kappa = \kappa(u)$ expressed by (possibly formal) series such that $h \cdot \nabla\kappa = 0$); as a consequence, its only symmetry (including nonlinear and possibly formal ones) having Au as a linear part is just $g_A = Au$ (let us recall that multiplying a symmetry by a constant of motion one obtains another symmetry). Then the coordinate transformation taking (1) into (7) transforms the symmetry g according to

$$g = Au + G \to g_A = Au.$$

This means that condition A is satisfied by this transformation of the symmetry. Thus, there is a normalizing transformation which is convergent, and one can easily conclude that under this transformation the DS is also transformed into NF.

Theorem 2. Instead of (i) in theorem 1 assume that the DS (1) admits ℓ ($\geqslant 1$) analytic symmetries $g_j = B_j u + G_j(u)$, where the matrices B_j ($\neq 0$) are linearly independent (and such that no linear combination is proportional to f), and where ℓ is precisely the number of the linearly independent linear symmetries admitted by the DS once in NF. Then, with the condition (ii) as in theorem 1, the same conclusion holds.

Sketch of the proof. According to a general property of NFs (see [3, 12]), the linear fields $B_j u$ are (linear) symmetries for the NF. Together with Au, we then have $\ell + 1$ symmetries for the NF; therefore, there must be one symmetry having just Au as a linear part, and the argument proceeds along similar lines as for theorem 1.

Notice that it can easily be seen that if the DS has dimension $n = 2$, the assumption (ii) in the above theorems is automatically satisfied, and then one re-obtains the Bruno and Walcher result which follows.

Corollary [8]. If a two-dimensional DS admits an analytic symmetry, it can be normalized by a convergent transformation.

We now give a quite general example, which can be interesting both for its possible applications to Hamiltonian DSs, and for illustrating the role played by the presence of symmetries in ensuring the convergence of the normalizing transformations.

Let $n = 2m$ and, putting $u \equiv (x_1, \ldots, x_m, y_1, \ldots, y_m) \in R^{2m}$, assume that a Lie group Γ acts 'diagonally' on both the m-dimensional spaces of the vectors x and y through the same linear representation $\mathcal{D}$, i.e. $x \to x' = \mathcal{D}x$, $y \to y' = \mathcal{D}y$, where $\mathcal{D}$ is an absolutely irreducible representation (i.e. the only matrix commuting with $\mathcal{D}$ is a multiple of the identity). Consider then a DS of the following form

$$\dot{u} = f(u) = Au + F(u) \tag{8}$$

where

$$A = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{8'}$$

and assume that $F(u)$ admits the symmetries $B_l u$, where B_l are the matrix representatives in the direct sum $\mathcal{D} \oplus \mathcal{D}$ of the Lie generators of this group Γ (the linear part Au fulfils this symmetry requirement, so that the full DS (8) admits this symmetry). For example, if the DS is

$$\dot{u} = Au + p(u)u \tag{9}$$

the scalar function p must depend only on the quantities $\rho_a = \rho_a(u)$ which are invariant under $\mathcal{D} \oplus \mathcal{D}$ (e.g. in the case $m = 3$, $\Gamma = SO(3)$ and $\mathcal{D}$ its fundamental representation, these are given by $x^2 = (x, x)$, $y^2 = (y, y)$, $x \cdot y = (x, y)$ where the parentheses stand for the scalar product in R^3). The NF of the DS (8) or (9) must admit the linear symmetries $B_l u$ and Au (see [3, 12]), then it is easy to see that it must take the form

$$\dot{u} = Au + \alpha Au + \mu u \tag{10}$$

where α and μ are functions of the quantities invariant under all these symmetries (e.g. of $r^2 = x^2 + y^2$ only, in the $SO(3)$ case). Assume now (cf the example in [9]) that in DS (9) the function p is a homogeneous polynomial of degree $2k$ built up with the quantities ρ_a. It is easy to verify that the following vector function (with vanishing linear part)

$$g = (x^2 + y^2)^k u \tag{11}$$

generates a non-trivial analytic symmetry for DS (9), and that there are no common constants of motion for (10), as required by (ii) in the above theorems. Therefore, the convergence of the normalizing transformation is ensured by theorem 1. It is important to notice that if our problem did not possess the symmetry Γ, the NF (10) would contain many other terms in its right-hand side, and that it is precisely the presence of the symmetry which forces the NF to contain only A and the identity and, therefore, allows us to apply the argument concerning the constants of motion. This seems to confirm the conjecture [8,13] that the presence of a 'sufficient' number of symmetries may be an essential requirement in order to guarantee the convergence of a normalizing transformation.

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